Centralizer of braids and Fibonacci numbers*

Usman Ali, Azeem Haider

Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakaria University, Multan, Pakistan. E-mail: {*uali, azeemhaider*}*@bzu.edu.pk*

Abstract

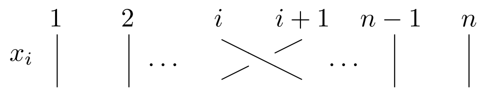
The paper encloses computation of simple centralizer of simple braids and their connection with Fibonacci numbers. Planarity of some commuting graphs is also discussed in the last section.

1 Introduction

The set of *positive n-braids* $\mathcal{MB}_n$ is defined by the classical presentation[1]:

$$\mathcal{MB}_n = \left\langle x_1, x_2, \cdots, x_{n-1} : \begin{array}{l} x_{i+1}\, x_i\, x_{i+1} \;=\; x_i\, x_{i+1}\, x_i \\ x_i\, x_j \;=\; x_j\, x_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle . \qquad (1)$$

In other words, a positive braid α is a word in the set of generators $\{x_1, x_2, \ldots, x_{n-1}\}$:



such that $\alpha \in \{x_{i_1}^{r_1} \cdots x_{i_k}^{r_k} : r_1, \ldots, r_k \geq 0\}$ and two words are equivalent if they can be transferred into each other by finitely many applications of relations given in (1), for example $x_4 x_3^2 x_2 x_3$ and $x_2 x_4 x_3 x_2^2$ are equivalent in $\mathcal{MB}_5$. The braid monoids $\mathcal{MB}_n$ are embedded in their corresponding braid groups $\mathcal{B}_n$, defined by the same presentation (1) (see [11]). A *simple braid* contains a letter x_i at most once (see [2]) and was shown in [3] that the number of simple braids in $\mathcal{SB}_n$ is Fibonacci number F_{2n-1}, where $(F_0, F_1, F_2, F_3, F_4, F_5 \ldots) = (0, 1, 1, 2, 3, 5, \ldots)$.

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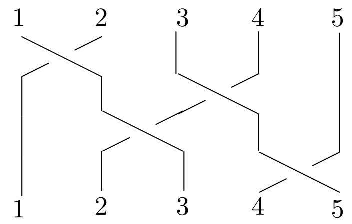
Corresponding author: uali@bzu.edu.pk

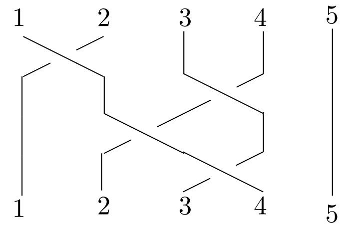
Three kinds of divisors of $\beta \in \mathcal{MB}_n$ are defined: divisors of β $(\gamma|\beta)$, $\mathrm{Div}(\beta) = \{\gamma \in \mathcal{MB}_n : \text{ there exist}\, \delta, \varepsilon \in \mathcal{MB}_n, \beta = \delta\gamma\varepsilon\}$; left divisors of β $(\gamma|_L\beta)$, $\mathrm{Div}_L(\beta) = \{\gamma \in \mathcal{MB}_n : \text{ there exists } \varepsilon \in \mathcal{MB}_n, \beta = \gamma\varepsilon\}$; and right divisors of β $(\gamma|_R\beta)$, $\mathrm{Div}_R(\beta) = \{\gamma \in \mathcal{MB}_n : \text{ there exists } \delta \in \mathcal{MB}_n, \beta = \delta\gamma\}$. Clearly $\mathrm{Div}_L(\beta) \cup \mathrm{Div}_R(\beta) \subseteq \mathrm{Div}(\beta)$.

The set $\mathcal{SB}_n$ is a proper subset of $\mathrm{Div}(\Delta_n)$, where

$$\Delta_n = x_1(x_2x_1)\cdots(x_{n-1}x_{n-2}\cdots x_2x_1)$$

is the *Garside braid* (see [2] for more details). The braid $x_1x_3x_2x_4$ is simple while the braid $x_1x_3x_2x_3$ is a non-simple divisor of Δ_n.





$\beta = x_1x_3x_2x_4$ $\qquad$ $\alpha = x_1x_3x_2x_3$

There is a canonical group homomorphism onto the symmetric group $\pi : \mathcal{B}_n \to \Sigma_n$; for example

$$\pi(\alpha) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}.$$

The restriction of π to $\mathrm{Div}(\Delta_n)$ is a bijection (see [9], chapter 9).

The symmetric group Σ_n also admits the following presentation in the Coxeter generators:

$$\Sigma_n = \left\langle s_1, s_2, \cdots, s_{n-1} : \begin{array}{l} s_i^2 = 1 \\ s_{i+1}\, s_i\, s_{i+1} \; = \; s_i\, s_{i+1}\, s_i \\ s_i\, s_j \; = \; s_j\, s_i \text{ for } |i-j| \geq 2 \end{array} \right\rangle. \tag{2}$$

The homomorphism π can be defined by $\pi(x_i) = s_i$. The image $\pi(\mathcal{SB}_n) = S\Sigma_n$ is called the set of simple permutations in Σ_n (see [4]).

The braid monoids $\mathcal{MB}_n$ satisfy left and right cancellation laws, i.e, $\beta\gamma = \beta\delta \Rightarrow \gamma = \delta$ and $\gamma\beta = \delta\beta \Rightarrow \gamma = \delta$ (see [11]). The monoid $\mathcal{MB}_n$ is embedded in $\mathcal{MB}_{n+1}$ and consequently, $\alpha, \beta \in \mathcal{MB}_n$ commutes in $\mathcal{MB}_{n+1}$ if and only if α and β commutes in $\mathcal{MB}_n$.

Definition 1.1. *The simple centralizer of* $\beta \in \mathcal{SB}_n$ *is the set* $C_n(\beta) = \{\gamma \in \mathcal{SB}_n : \beta\gamma = \gamma\beta\}$, *i.e, the intersection of centralizer of* β *in* $\mathcal{B}_n$ *with* $\mathcal{SB}_n$. *The cardinality of* $C_n(\beta)$ *is denoted by* $c_n(\beta)$.

Our first result completely describes the structure of $C_n(x_i)$ for $1 \leq i \leq n-1$.

Theorem 1.2. *For any* $x_i \in \mathcal{SB}_n$, *we have*

$$C_n(x_i) = \{\beta \in \mathcal{SB}_n : x_j \nmid \beta \,\text{for}\ \mid j - i \mid = 1\}.$$

For $x_{n-1}|\beta$ (where $n-1 = \max\{i : x_i \mid \beta\}$), our second result describes the structure of $C_{n+m}(\beta)$ in terms of $C_n(\beta)$.

Theorem 1.3. *If* $\beta \in \mathcal{SB}_n$ *and* $x_{n-1}|\beta$, *then* $\gamma \in C_{n+m}(\beta)$ *if and only if* $\gamma = \gamma_1\gamma_2$ *where* $m \geq 1$, $\gamma_1 \in C_n(\beta)$, *and* $\gamma_2 \in E_n = \{\alpha \in \mathcal{SB}_{n+m} : x_j \nmid \alpha \text{ for all } j \leq n\}$.

It is clear that the sets E_n and $\mathcal{SB}_m$ have the same number of elements and consequently the following holds.

Corollary 1.4. *If* $\beta \in \mathcal{SB}_n$ *and* $x_{n-1}|\beta$, *then* $c_{n+m}(\beta) = c_n(\beta)F_{2m-1}$ *for all* $m \geq 1$.

The value of $c_{n+m}(x_{n-1})$ is given by the following proposition.

Proposition 1.5. *a)* $c_{2+m}(x_1) = 2F_{2m-1}$ *and*
b) $c_{n+m}(x_{n-1}) = 2F_{2n-5}.F_{2m-1}\ \forall\ n \geq 3$.

In the next section proofs of Theorem 1.2, Theorem 1.3, and Proposition 1.5 are given.

An algorithm is given in [10] to compute the centralizer of an arbitrary braid $\beta \in \mathcal{B}_n$. In the same paper, simple elements are defined as divisors of the Garside braid Δ_n (see definition 4 in [10]).

Definition 1.6. *([13], page 248) A graph is said to be planar if it can be embedded in a plane. Otherwise it is non-planar.*

Definition 1.7. *([5, 7, 12]) A commuting graph* $\Gamma(H)$ *associated to a group* G *and a finite subset* H *of* G *is a graph whose vertices are the elements of* $H \setminus \{e\}$ *and there is an edge between* g *and* h *if and only if* $g \neq h$ *and* $gh = hg$.

In the last section, we discussed the planarity of three graphs: $\Gamma(\mathcal{SB}_n)$, $\Gamma(S\Sigma_n)$, and $\Gamma(\Sigma_n)$. The graph $\Gamma(\Sigma_n)$ was analyzed in [12].

Proposition 1.8. *a)* $\Gamma(\mathcal{SB}_n)$ *is planar if and only if* $n \leq 5$ *and*
b) $\Gamma(S\Sigma_n)$ *and* $\Gamma(\Sigma_n)$ *are planar if and only if* $n \leq 4$.

The graph $\Gamma(\mathcal{SB}_n)$ is a proper subgraph of $\Gamma(S\Sigma_n)$ for all $n \geq 3$; for example x_1x_2 and x_2x_1 does not commute in $\mathcal{SB}_n$, but their images $\pi(x_1x_2)$ and $\pi(x_2x_1)$ commutes in $S\Sigma_n$.

2 Simple Centralizer

Lemma 2.1. *([11], Theorems* H, K) *a) If $x_i|_L\beta$ and $x_j|_L\beta$ for $\beta \in \mathcal{MB}_n$ and $i \neq j$ then:*
i) $x_ix_j|_L\beta$ *for* $|i-j| \geq 2$ *and*
ii) $x_ix_jx_i|_L\beta$ *for* $|i-j| = 1$.

b) If $x_i|_R\beta$ and $x_j|_R\beta$ for $\beta \in \mathcal{MB}_n$ and $i \neq j$ then:
i) $x_ix_j|_R\beta$ *for* $|i-j| \geq 2$ *and*
ii) $x_ix_jx_i|_R\beta$ *for* $|i-j| = 1$.

Lemma 2.2. *For $\beta, \gamma \in \mathcal{MB}_n$ we have*
a) if $x_{n-1}|_L\beta\gamma$ and $x_{n-1} \nmid \beta$ then $x_{n-1}|_L\gamma$,
b) if $x_{n-1}|_R\gamma\beta$ and $x_{n-1} \nmid \beta$ then $x_{n-1}|_R\gamma$.

Proof. a) The induction on the length $|\beta|$ of β starts at $|\beta| = 1$: for $\beta = x_i$, Lemma 2.1a)i) implies $x_i\gamma = x_ix_{n-1}\delta$ for some $\delta \in \mathcal{MB}_n$. By cancellation property, $x_{n-1}|_L\gamma$. For $|\beta|$, suppose $\beta = x_i\beta_1$ implies $x_{n-1}|_L\beta_1\gamma$. By induction hypothesis we have $x_{n-1}|_L\gamma$.
b) The proof is symmetric to a). □

Proof of the Theorem 1.2: If $x_j \nmid \beta$ then, by the presentation (1), $\beta x_i = x_i\beta$ and $\beta \in C_n(x_i)$. Conversely, apply induction on the length $|\beta|$ of β. For $|\beta| = 1$, $x_ix_j = x_jx_i$ implies $|i-j| \geq 2$ or $i = j$ by the presentation (1). For $|\beta|$, suppose on contrary, i.e., $\beta \in C_n(x_i)$ and $x_j \mid \beta$. Let $\beta = x_k\beta_1$ and consider the two cases: a) $k = j$ and b) $k \neq j$. For a), by simplicity of β we have $x_j \nmid \beta_1x_i$ and $x_j\beta_1x_i = x_ix_j\beta_1$ but then, by Lemma 2.1, $x_j\beta_1x_i = x_jx_ix_j\beta_2$ for some positive braid β_2. Hence $\beta_1x_i = x_ix_j\beta_2$, which contradicts that $x_j \nmid \beta_1x_i$. For b), $x_j \mid \beta$ and $x_k\beta_1x_i = x_ix_k\beta_1 = x_kx_i\beta_1$ implies $\beta_1x_i = x_i\beta_1$ which contradicts the induction hypotheses. □

Lemma 2.3. *If $\beta \in \mathcal{SB}_n$ and $x_{n-1}|\beta$, then either $x_{n-1}|_R\beta$ or $x_{n-1}|_L\beta$.*

Proof. Since β contains x_{n-2} at most once so x_{n-2} must be either on the right of x_{n-1} or on the left of it in β. If x_{n-2} is on the right then, by the presentation (1), x_{n-1} can be moved to the left most in β. Similarly x_{n-1} can be moved to the right most in β, if x_{n-2} is on the left. □

Lemma 2.4. *If $\beta \in \mathcal{SB}_n$ and $\alpha \in \mathcal{SB}_{n+1}$ such that $x_{n-1}|\beta$ and $x_n|\alpha$ then $\beta\alpha \neq \alpha\beta$.*

Proof. Suppose on contrary. By Lemma 2.3, the following four cases have to be dealt with:
a) $\beta = x_{n-1}\beta_1$, $\alpha = x_n\alpha_1$; b) $\beta = x_{n-1}\beta_1$, $\alpha = \alpha_2x_n$; c) $\beta = \beta_2x_{n-1}$,

$\alpha = x_n\alpha_1$; and d) $\beta = \beta_2 x_{n-1}$, $\alpha = \alpha_2 x_n$, where $\beta_1, \beta_2 \in \mathcal{SB}_{n-1}$ and $\alpha_1, \alpha_2 \in \mathcal{SB}_n$.

For a), $x_{n-1}\beta_1 x_n \alpha_1 = x_n \alpha_1 x_{n-1}\beta_1$. By Theorem 1.2, $x_{n-1}x_n\beta_1\alpha_1 = x_n\alpha_1 x_{n-1}\beta_1$. By Lemma 2.1 and the cancellation property, $\alpha_1 x_{n-1}\beta_1 = x_{n-1}x_n\delta$ for some $\delta \in \mathcal{MB}_{n+1}$ which means $x_n|\alpha_1$ or $x_n|\beta_1$, a contradiction.

For b),

$$x_{n-1}\beta_1\alpha_2 x_n = \alpha_2 x_n x_{n-1}\beta_1. \tag{3}$$

By Lemma 2.2b, $\alpha_2 x_n x_{n-1} = \delta x_n$ for some $\delta \in \mathcal{MB}_{n+1}$. By Lemma 2.1b)ii) and the cancellation property, $\alpha_2 = \delta_1 x_{n-1}$ for some $\delta_1 \in \mathcal{MB}_{n+1}$. Using equation (3), $x_{n-1}\beta_1\alpha_2 x_n = \delta_1 x_{n-1}x_n x_{n-1}\beta_1 = \delta_1 x_n x_{n-1} x_n \beta_1 = \delta_1 x_n x_{n-1}\beta_1 x_n$. We get $x_{n-1}\beta_1\alpha_2 = \delta_1 x_n x_{n-1}\beta_1$ which means $x_n|\alpha_2$ or $x_n|\beta_1$, a contradiction.

The proofs of the cases c) and d) are symmetric to a) and b).

$\square$

The following holds in an arbitrary monoid $\mathcal{M}$ with the cancellation law.

Lemma 2.5. *If $\beta(\gamma_1\gamma_2) = (\gamma_1\gamma_2)\beta$ and $\beta\gamma_1 = \gamma_1\beta$ or $\beta\gamma_2 = \gamma_2\beta$, then $\beta\gamma_2 = \gamma_2\beta$ or $\beta\gamma_1 = \gamma_1\beta$ respectively.*

Lemma 2.6. *If $\beta \in \mathcal{SB}_n$ and $x_{n-1}|\beta$, then $x_n \nmid \gamma$ for any $\gamma \in C_{n+m}(\beta)$, where $m \geq 1$.*

Proof. By Lemma 2.3 all the divisors x_j, $j \geq n+1$ can be written at the beginning or at the end, i.e, $\gamma = \eta\rho\mu$ such that $\rho \in \mathcal{SB}_{n+1}$, $x_j \nmid \eta$ and $x_j \nmid \mu$ for $j \leq n$. Since $\beta\eta = \eta\beta$ and $\beta\mu = \mu\beta$, so by Lemma 2.5, we find $\beta\rho = \rho\beta$. By Lemma 2.4, $x_n \nmid \rho$. $\square$

Proof of the Theorem 1.3: Follows from Lemma 2.6 and the presentation (1).

Lemma 2.7. *a) $c_2(x_1) = 2$ and*
b) $c_n(x_{n-1}) = 2F_{2n-5}$ $\forall$ $n \geq 3$.

Proof. a) Clearly $C_n(x_1) = \mathcal{SB}_2 = \{e, x_1\}$.

b) Since $\mathcal{SB}_{n-2}.x_{n-1} = x_{n-1}.\mathcal{SB}_{n-2}$, so by Theorem 1.2,

$$C_n(x_{n-1}) = \mathcal{SB}_{n-2} \amalg x_{n-1}.\mathcal{SB}_{n-2}.$$

$\square$

Proof of the Proposition1.5: By Corollary 1.4 and Lemma 2.7. $\square$

3 Commuting graphs

Clearly the degree of $\beta \in \mathcal{SB}_n$ in $\Gamma(\mathcal{SB}_n)$ is equal to $c_n(\beta) - 2$: we minus e and β from $C_n(\beta)$ in order to make the vertices non-central and avoid loop in $\Gamma(\mathcal{SB}_n)$. The graph $\Gamma(\mathcal{SB}_{n-1})$ has a canonical embedding in $\Gamma(\mathcal{SB}_n)$. Similarly, The graphs $\Gamma(S\Sigma_{n-1})$ and $\Gamma(\Sigma_{n-1})$ are embedded in $\Gamma(S\Sigma_n)$ and $\Gamma(\Sigma_n)$ respectively.

The canonical form for a positive braid (the smallest element in the equivalence class of a braid) was introduced in [8, 6] to solve the word problem in $\mathcal{MB}_n$. In order to generate the $\Gamma(\mathcal{SB}_5)$, we used these canonical forms, Theorem 1.2, Lemma 2.4, and Lemma 2.5.

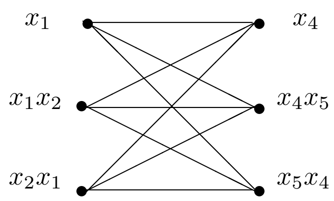
Examples 3.1. *1)* $\alpha = x_1x_3x_2$ *and* $\beta = x_2x_3$ *does not commute because the canonical form of* $\alpha\beta$ *and* $\beta\alpha$ *is* $x_1x_3x_2^2x_3$ *and* $x_2x_1x_3^2x_2$ *respectively.*

2) $\alpha = x_1x_3$ *and* $\beta = x_2x_4x_3$ *does not commute by Lemma 2.4.*

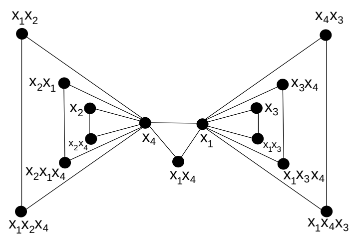
3) $\alpha = x_1x_4$ *and* $\beta = x_2x_1$ *does not commute by Lemma 2.5.*

Lemma 3.2. (*[13], page 250, example* 7.1.5.) *The complete bipartite graph* $K_{3,3}$ *and the complete graph* K_5 *are non-planar.*

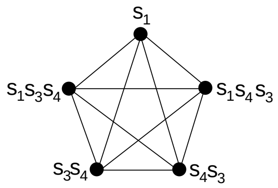
Proof of the Proposition 1.8: a) The graph $\Gamma(\mathcal{SB}_6)$ is non-planar (by Lemma 3.2) because it contains $K_{3,3}$ as a subgraph:



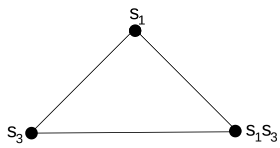
There are 33 non-central elements in $\mathcal{SB}_5$. The connected component of $\Gamma(\mathcal{SB}_5)$ is given in the following diagram and the rest of the vertices are isolated



b) The graph $\Gamma(S\Sigma_5)$ is non-planar (by Lemma 3.2) because it contains K_5 as a subgraph:



It is easy to check that $\Gamma(\Sigma_4)$ is planar: $\Gamma(\Sigma_4)$ contains the following cycle and the rest of vertices have degrees at most one.



$\square$

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